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Generalised Neighbour Designs in Circular Blocks with Group-Divisible Association Scheme for Correlated Observations

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SUMMARY

In an m class Generalised Neighbour Balanced (GNm) design in circular blocks, any two treatments can occur together as neighbours in the design λ_i ($i = 1, 2, \ldots, m$) number of times (Misra *et al.*1991; Iqbal *et al*. 2012; Hamad 2014 etc.). Literature survey reveals that in existing GNm designs no association scheme is involved for any pair treatments as neighbours. In the present paper a new series of GN_2 designs from Group Divisible (GD) PBIB designs (Clatworthy 1973) has been developed. The series of designs also has a GD association scheme (Singular or Semi-regular or Regular) for the treatments as neighbours. The concept of merging of GD PBIB designs and the above mentioned GN_2 designs with correlated observations will develop a new series of Group divisible Generalised neighbour balanced PBIB designs with two associate classes (GDGNPBIB). The definitions, properties and structural characteristics of GDGNPBIB are also developed in paper. The blocks used in these designs are considered as circular using border plot concept (Azais *et al*. 1993; Bailey 2003; Jaggi *et al*. 2007; Majumder *et al*. 2015 etc.). The presence of correlation in the form of neighbour effects among the plots in agricultural experiments is used as given in Kiefer and Wynn (1981), Gill and Shukla (1985). The main advantage of GDGNPBIB designs with correlated observations is that the analyses of such designs are identical with GD PBIB designs. It has observed that any GD PBIB design with $k = 3$ will be a GDGNPBIB design in circular blocks. Moreover, a series of GDGNPBIB designs in circular blocks has been listed for $k = 4$. The efficiency values (A & D) of listed designs are presented for different values of ρ (-0.5 to +0.5).

Keywords: GN_2 designs, Group Divisible Association scheme, PBIB designs, GDGNPBIB designs in circular blocks, Correlated observations.

1. INTRODUCTION

Rees (1967) introduced the concept of neighbour balanced design in circular blocks in serology. Since then, the concept of construction of neighbour balanced design has become an important topic in statistics and its optimality criteria been studied in a much broader way.

In a design with Circular blocks, positions of treatments in experimental plots are considered in 2-dimensions (Left and Right). Let D (v, b, k) be a block design D with v treatments in b blocks of size k. In design D, if treatment **a** (= 1, 2,…, v) is on i^{th} plot (= 1, 2, …,k) in j^{th} block (= 1,2,…,b), then the treatment **a**′ on $(i+1)^{th}$ plot (mod k) is regarded as the right-neighbour of treatment **a** in that j^{th} block. On the contrary, the treatment on $(i-1)^{th}$ plot (mod k), is considered as the left-neighbour of treatment **a** in j^{th} block . Many authors (e.g. Azais *et al*. 1993; Bailey 2003; Tomar *et al*. 2005; Jaggi *et al*. 2006; Jaggi *et al*. 2007; Tomar and Jaggi 2007; Pateria *et al*. 2011; Bhowmik *et al*. 2012, Majumder *et al*. 2015 etc.) used the concept of border plots at the starting and at the end positions of a block. Here, the treatment in the last plot (or at right end) of a block has been placed in the initial border plot and the treatment in first plot (or at left end) of a block has been placed in the last border plot of the block. However, the border plot treatment effects are not considered in the analysis of the design. Thus, a circular block can also be used as a linear

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block by using border plots at both ends of a linear block. Therefore, each treatment in a circular block has two neighbours, one on left and another on right.

If each treatment has every other treatment as neighbour equal number of times then the design is said to be a neighbour balanced (NB) design (Iqbal *et al.* 2012). Neighbour balance (NB) is desirable if it is known or thought that the effect of a plot is influenced by its neighbouring plots, in such cases nearest neighbour analysis is considered to be more efficient than classical analysis methods (Wilkinson *et al.* 1983; Gill and Shukla 1985). It also provides protection against the effects of correlated observations or potentially unknown trends highly correlated with plot positions within a block (Keifer and Wynn 1981; Cheng 1983; Jackroux 1998; Bailey and Druilhet 2004; Ahmed *et al*. 2011 and Sahu and Majumder 2012). But the condition in the definition of NB for a design cannot be met, always. In such situations, Misra (1988) and Misra *et al*. (1991) proposed the concept of m class Generalised Neighbour Balanced (GNm) designs as any two treatments can occur together as neighbours in the design λ_i ($i = 1, 2, \ldots, m$) number of times. When λ_i = a constant $\forall\ i$ (=1, 2,…, m) the design becomes NB. Till then, several authors (Mishra 1988; Mishra and Chaure 1989; Mishra *et al*. 1991; Jaggi *et al*. 2006; Ahmed *et al.* 2009; Iqbal *et al*. 2012 etc.) constructed different series of GNm designs as GN_2 or GN_3 etc. All the series of the above designs are not identical in respect of neighbouring pairs. Actually, no particular association scheme is involved in the present literature of GNm designs. Recently, Hamad (2014) generated a series GN_2 and GN_3 designs (binary and non-binary) as partially neighbour balanced designs in circular blocks without any particular association scheme.

It is well established fact that apart from BIB designs, PBIB designs are the most desirable in the field of non-orthogonal block designs block designs due to its well defined association schemes. The present piece of investigation aims to construct GN_2 designs with higher efficiency values having Group Divisible association scheme among the treatments as neighbours using GDPBIB designs. The developed designs are basically PBIB designs (v, b, r, k, n_i and λ_{1i}) with group divisible association scheme. In these designs every blocks are circular in nature keeping border plots on both end sides of plots (Azais *et al.* 1993 and Tomar *et al.* 2005) and every treatment having immediate neighbors λ_{2i} (i = 1 & 2) number of times (Misra 1988). These neighbors are also following the similar association scheme of GDPBIB designs. Therefore, the developed designs are called as GDGNPBIB designs with parameters v, b, r, k, n_i, λ_{1i} and λ_{2i}. The analysis and properties of such designs are identical to common GD designs even if the observations are correlated which surely provide advantages to the programmers and other users.

1.1 Useful Definitions and Model

Following are some useful definitions associated with nieghbour balanced block designs under correlated observations:

Definition 1.1: A block of treatments with border plots will be **circular** if a treatment in the left border is same as the treatment in the right end inner plot as well as if a treatment in the right border is same as the treatment in the left end inner plot. If all the blocks of a design are circular, then it is a **circular design**.

Definition 1.2: A block design in circular blocks is **neighbour balanced** (NB) if every treatment occurs equally often with every other treatment as neighbours.

Definition 1.3: A block design in circular blocks will be **two class Generalized Neighbour Balanced** (GN_2) design, if any two treatments can occur together as neighbours in the design λ_i (i = 1 & 2) number of times.

Definition 1.4: Two class Generalized Neighbour Balanced (GN_2) design in circular blocks with two class association scheme will be **two class Generalized Neighbour Partially Balanced Incomplete Block** (GN_2PBIB) design with an arrangement of v treatments in b circular blocks of size k ($k < v$) when

1. Any treatment occurs in r circular blocks.
2. Any pair of treatments occurs together in λ_{1i} (i = 1 & 2) circular blocks and same pair treatments occurs together as neighbour in λ_{2i} (i = 1 & 2) circular blocks.
3. Any treatment has exactly n_i (i = 1 & 2) number of i^{th} associate treatments.
4. If the treatments α and β are mutually i^{th} associates in the association scheme, then α and β occur together in λ_{1i} (i = 1 & 2) blocks

as well as α and β occur together as neighbour in λ_{2i} (i = 1 & 2) blocks, where the integers λ_{1i} and λ_{2i} do not depend on the pair α and β and the treatments α and β are mutually i^{th} associates (i = 1 & 2).

In a GN_2PBIB design, either all λ_{1i}'s and λ_{2i}'s are unequal, or all λ_{1i}'s are unequal but λ_{2i}'s are equal or all λ_{1i}'s are equal but λ_{2i}'s are unequal. Further, the association matrices (P) with the elements $p_{jk}^{\ i}$ are similar to PBIB designs.

The integers v, b, r, k, n_i, λ_{1i} and λ_{2i} are the parameters of the GN_2PBIB design. Therefore, the parametric relations of a GN_2PBIB design are:

1. $vr = bk$,
2. $\sum_i n_i = v-1$,
3. $\sum_i n_i\lambda_{1i} = r(k-1)$ and
4. $\sum_i n_i\lambda_{2i} = 2r$.

The proofs of the first three relations are established for any PBIB design. In a design with circular blocks any treatment in any plot of the design has two neighbours. In the design each treatment is occurring r number of times. Therefore, the total number of neighbours of any treatment in the design will be $2r$ which is nothing but $\sum_i n_i\lambda_{2i}$.

Let us consider a class of GN_2PBIB designs with v treatments, b blocks of sizes k ($<v$) with correlated observations.

Let $\mathbf{Y_{ij}}$ be the response from the i^{th} plot in the j^{th} block (i = 1, 2,.., k, j = 1, 2, …, b). Moreover, the layout includes border plots at both ends of every block i.e., at 0^{th} and $(k+1)^{th}$ position whose effects are not taken under analysis. In what follows, we only consider a fixed effects additive model for analyzing a block design having correlated observations. It is also to be mentioned that the effects of the neighbouring plots are not taken under consideration in the present model as the main aim of the present study is to simplify the analysis procedure of PBIB designs with correlated observations whose neighbouring pairs are also following the similar association scheme.

$$\mathbf{Y = \mu 1 + X\tau + Z\beta + \varepsilon}, \quad (1.1)$$

where, $\mathbf{Y}$ is a $n \times 1$ vector of observations, $\boldsymbol{\mu}$ is a general mean, $\mathbf{1}$ is a $n \times 1$ vector of ones, $\mathbf{X}$ is a $n \times v$ incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ is a $v \times 1$ vector of treatment effects, $\mathbf{Z}$ is a $n \times b$ incidence matrix of observations versus blocks, $\boldsymbol{\beta}$ is a $b \times 1$ vector of block effects and $\boldsymbol{\varepsilon}$ is a $n \times 1$ vector of random errors. According to Gill and Shukla (1985), $\boldsymbol{\varepsilon}$ be the error terms independently and normally distributed with mean zero and a variance and covariance matrix be $\mathbf{V}$, such that $\mathbf{V}^{-1} = \sigma_e^{-2}\,\mathbf{I}_b \otimes \mathbf{W}_k$ ($\mathbf{I_b}$ is an identity matrix of order b, $\otimes$ denotes the kronecker product and $\mathbf{W}_k$ is the correlation matrix of k observations within a block). Assuming the NN_1 model of Kiefer and Wynn (1981), that there is no correlation among the observations between the blocks and correlation structure between plots within a circular block to be the same in each block and the structure of $\mathbf{W}_k$ will be,

$$\mathbf{W}_k = \begin{bmatrix} 1 & \rho & 0 & \cdots & \rho \\ \rho & 1 & \rho & 0 & 0 \\ \vdots & \rho & \ddots & \rho & \vdots \\ 0 & \cdots & \rho & 1 & \rho \\ \rho & 0 & \cdots & \rho & 1 \end{bmatrix} \quad (1.2)$$

where ρ ($-1 \leq \rho \leq +1$) is the correlation coefficient between the neighbouring plots in a block. Let us consider the above design set-up in the line of Gill and Shukla (1985). Then, the information matrix ($\mathbf{C}$ matrix) for estimating the treatment effects having correlated observations estimated by generalized least squares will be as follows:

$$\mathbf{C} = \left[\mathrm{X'V^{-1}X}\right] - \left[\mathrm{X'V^{-1}Z}\left(\mathrm{Z'V^{-1}Z}\right)^{-1}\mathrm{Z'V^{-1}X}\right] \quad (1.3)$$

The above $\mathbf{C}$ matrix (1.3) for estimating the effect of treatments in a block design is symmetric, non-negative definite with zero row and column sums.

Let us consider only 2 class Generalized Neighbour Partially Balanced Incomplete Block (GN_2PBIB) designs with parameters v, b, r, k, n_1, n_2, λ_{11}, λ_{12}, λ_{21} and λ_{22} for the results in the next section.

2. IMPORTANT RESULTS ON GN_2PBIB DESIGNS FOR CORRELATED OBSERVATIONS

According to model (1.1) with correlated observations as given in 1.2, the matrix $X'V^{-1}Z$ will be considered as the treatment vs. block incidence

matrix of order $v \times b$. Here, it should be noted that the structure of the matrix $[(X'V^{-1}Z) \times (Z'V^{-1}X)]$ (or $N^*N^{*\prime}$) is identical to structure of the NN' of any two associate class PBIB designs, where N is the treatment vs. block incidence matrix of order $v \times b$ of the PBIB design. Thus, the diagonal elements of the matrix $N^*N^{*\prime}$ are all equal to $r(1+2\rho)^2$ and in each row of $N^*N^{*\prime}$ there are precisely n_1 positions filled by $\lambda_{11}(1+2\rho)^2$ and precisely n_2 positions filled by $\lambda_{12}(1+2\rho)^2$. Again, the diagonal elements of the matrix $X'V^{-1}X$ (or $r^{*\delta}$) will be r and in each row of $r^{*\delta}$ there are precisely n_1 positions filled by $\rho\lambda_{21}$ and precisely n_2 positions filled by $\rho\lambda_{22}$. In the design, the matrix $Z'V^{-1}Z$ (or $k^{*\delta}$) = $[k(1+2\rho) \times \mathbf{I}]$, where $\mathbf{I}$ is an identity matrix of order b.

Lemma 2.1: The row or column sum of the matrix $[(X'V^{-1}Z) \times (Z'V^{-1}X)]$ of the above GNPBIB2 design will be $rk(1+2\rho)^2$ and det. of the matrix $[(X'V^{-1}Z) \times (Z'V^{-1}X)]$ will be $rk(1+2\rho)^2\left[rk(1+2\rho)^2 - v\lambda_{12}(1+2\rho)^2\right]^{(n_1-1)}\left[r(1+2\rho)^2 - \lambda_{11}(1+2\rho)^2\right]^{(n_2-1)}$.

Proof. The result is straight forward following the steps given in Bose and Conor (1952) on the matrices developed by model 1.1 and 1.2.

Lemma 2.2: The diagonal elements of the C- matrix (1.3) of a GN_2PBIB design will be

$C_{ii} = \dfrac{r[k^2 - k(1+2\rho)]}{k^2}$ and in each row of C, there are precisely n_1 positions filled by

$C_{ij_1} = \dfrac{-\lambda_{11}k(1+2\rho) + k^2\rho\lambda_{21}}{k^2}$, $(j_1 = 1, 2,\ldots, n_1)$ and precisely n_2 positions filled by

$C_{ij_2} = \dfrac{-\lambda_{12}k(1+2\rho) + k^2\rho\lambda_{22}}{k^2}$, $(j_2 = 1, 2,\ldots, n_2)$.

Proof.: According to the model with correlated observations (Gill and Shukla 1985) the structure of C matrix has been given in 1.3. The proof is straight forward for the GN_2PBIB design with parameters v, b, r, k, n_1, n_2, λ_{11}, λ_{12}, λ_{21} and λ_{22}.

Hereinafter, considered only the Group Divisible GN_2PBIB designs called as GDGNPBIB designs.

2.1 GD two-Class Association Scheme

Let there be $v = mn$ treatments (m, n integer; $m \geq 2$, $n \geq 2$) arranged in rectangular array with m rows and n columns. On these mn treatments, we define a *Group-divisible* (GD) association scheme as: two treatments are first associates if they belong to the same row of the array and others are second associates (Clatworthy 1973).

The parameters of the GD association scheme are as follows

$v = mn$, $n_1 = n-1$, $n_2 = n(m-1)$,

$$P_1 = (p_{ij}^1) = \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix}$$

$$P_2 = (p_{ij}^2) = \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}$$

2.2 Group Divisible GN2PBIB(GDGNPBIB) Designs for Correlated Observations

A GN_2PBIB design is said to be Group divisible design (GDGNPBIB) if it is based on Group Divisible (GD) association scheme. Let $N^* = (X'V^{-1}Z)$ be the incidence matrix of the GDGNPBIB with parameters $v = mn$, b, r, k, $n_1 = n-1$, $n_2 = n(m-1)$, λ_{11}, λ_{12}, λ_{21} and λ_{22}. Then the eigen roots (θ_{1i}, $i = 0,1,2$) with the respective multiplicity α_i, ($i = 0,1,2$) of the matrix $N^*N^{*\prime}$ (Lemma 2.1) will be

$$\theta_{10} = rk(1+2\rho)^2, \quad \alpha_0 = 1; \tag{2.1}$$

$$\theta = (1+2\rho) - \frac{\left[\left\{(\lambda_{11} - \lambda_{12})\left(-\gamma + (-1)^i\sqrt{\Delta}\right) + (\lambda_{11} + \lambda_{12})\right\}(1+2\rho)\right]}{2}, \quad i=1,2; \tag{2.2}$$

$$\alpha_i = \frac{n_1 + n_2}{2} + (-1)^i\left[\frac{(n_1 - n_2) + \gamma(n_1 + n_2)}{2}\right], \quad i=1,2,$$

Where,

$\gamma = p_{12}^2 - p_{12}^1$, $\beta = p_{12}^1 + p_{12}^2$, $\Delta = \gamma^2 + 2\beta + 1$, $n_1 = n-1$ and $n_2 = n(m-1)$.

Since Lemma 2.1, $\mathbf{N}^*\mathbf{N}^{*\prime}$ is a non-negative definite, it is necessary that $\left[r(1+2\rho)^2\right] \geq \left[\lambda_{11}(1+2\rho)^2\right]$ and $\left[rk(1+2\rho)^2\right] \geq \left[v\lambda_{12}(1+2\rho)^2\right]$, which simply implies ($r \geq \lambda_{11}$) and ($rk \geq v\lambda_{12}$). Based on the eigen roots of $\mathbf{N}^*\mathbf{N}^{*\prime}$ or det. of $\mathbf{N}^*\mathbf{N}^{*\prime}$, GDGNPBIB designs can be divided into the following three types

Singular, if $r = \lambda_{11}$;

Semi-regular, if $r > \lambda_{11}$ and $rk = v\lambda_{11}$ and

Regular, if $r > \lambda_{11}$ and $rk > v\lambda_{11}$.

Similarly, for $(X'V^{-1}X)$ matrix of GDGNPBIB Design, the eigen roots (θ_{2i}, i= 0,1,2) with multiplicities α_i (i= 0,1,2) times will be

$$\theta_{20} = r + (n_1\lambda_{21} + n_2\lambda_{22})\rho = r(1+2\rho), \ \alpha_0 = 1; \quad (2.3)$$

$$\theta_{2i} = r - \frac{\left[\left\{(\lambda_{21} - \lambda_{22})\left(-\gamma + (-1)^i\sqrt{\Delta}\right) + (\lambda_{21} + \lambda_{22})\right\}\rho\right]}{2}, \quad i=1,2, \alpha_i; \quad (2.4)$$

$$\alpha_i = \frac{n_1 + n_2}{2} + (-1)^i\left[\frac{(n_1 - n_2) + \gamma(n_1 + n_2)}{2}\right], \quad i=1 \text{ and } 2.$$

Based on above results of (2.1 to 2.4), we have the following lemma.

Lemma 2.3: The eigen roots (values) of information matrix (C) of GDGNPBIB Design with parameters $v = mn$, b, r, k, $n_1 = n$-1, $n_2 = n(m$-1), λ_{11}, λ_{12}, λ_{21} and λ_{22} will be

$$\kappa_0 = \left[\theta_{20} - \frac{\theta_{10}}{k(1+2\rho)}\right] = 0 \text{ with multiplicity 1;}$$

$$\kappa_1 = \left[\theta_{21} - \frac{\theta_{11}}{k(1+2\rho)}\right] \text{with multiplicity } \alpha_1 \text{ and}$$

$$\kappa_2 = \left[\theta_{22} - \frac{\theta_{12}}{k(1+2\rho)}\right] \text{ with multiplicity } \alpha_2.$$

Proof: Results are straight forward following the results from 2.1 to 2.4.

3. METHOD OF CONSTRUCTION FOR GDGNPBIB DESIGNS

An extensive list of PBIB designs with two associate classes was prepared by Clatworthy (1973). The solutions of the listed GD (Singular [S], Semi Regular [SR] and Regular [R]) designs or the repetition of the listed designs of Clatworthy (1973) are used to construct GDGNPBIB Designs in circular blocks by either a little adjustment of positions of treatments in a block or by no adjustment.

Method 3.1: All the listed GD designs in Clatworthy (1973) with k = 3 are themselves a GDGNPBIB Designs in circular blocks with parameters [$v = mn$, b, r, k = 3, $n_1 = n$-1, $n_2 = n(m$-1), λ_{11}, λ_{12}, $\lambda_{21}(=\lambda_{11})$ and $\lambda_{22}(=\lambda_{12})$].

Proof: In a circular block with size 3, each treatment in a block has two neighbours. It is obvious that the occurrence of number of pairs of i^{th} (1st or 2nd) associate treatments in the design will be same as the occurrence of number of pairs of same i^{th} (1st or 2nd) associate treatments as neighbours in the design.

Example 3.1: Consider a Semi regular PBIB design (SR18, Clatworthy, 1973) with parameters v = 6 (=3x2), b = 4, r = 2, k = 3, n_1=1, n_2=4, λ_1=0, λ_2=1. The above PBIB design itself is a GDGNPBIB designs (v = 6, b = 4, r = 2, k = 3, m = 3, n = 2, n_1 = 1, n_2 = 4, λ_{11}=0, λ_{12}=1, λ_{21}(=0) and λ_{22}(=1)) in circular blocks, whose solution is given by:

3	1	2	3	1
6	1	5	6	1
6	2	4	6	2
5	3	4	5	3

The above GDGNPBIB design requires lesser number of blocks compared to the GN2 design (see number 2 of Table 1) of Iqbal, *et al*., (2012) for same v = 6 and k = 3.

Method 3.2: A series of GDGNPBIB Designs in circular blocks having parameters [$v = mn$, b, r, k ($\geq$ 4), $n_1 = n$-1, $n_2 = n(m$-1), λ_{11}, λ_{12}, λ_{21} and λ_{22}] can be constructed from the GD designs or the repetition of the designs of Clatworthy (1973) with parameters [$v = mn$, b, r, k ($\geq$ 4), $n_1 = n$-1, $n_2 = n(m$-1), λ_1 λ_2] after a little adjustment of positions of treatments in a block or without adjustment if the following conditions are satisfied.

1. For any two integers **s** and **t** (both should not be zero simultaneously), then $(sn_1 + tn_2) \mid 2r$.
2. $\sum_i n_i\lambda_{2i} = 2r$, i = 1, 2.

Proof: Let the 1st (or 2nd) associate treatments of a particular treatment be appeared as neighbour in the design **s** (or **t**) number of times, where **s** and **t** be either zero or any positive integer but both should not be zero simultaneously. The total number of neighbours in a

GDGNPBIB design with respect to any treatment will be 2r. This 2r should be divisible by the occurrences of $i^{th}(1^{st}$ or $2^{nd})$ associate treatments as neighbours. Thus the condition 1 is satisfied. The proof of condition 2 is already given in Definition 1.4.

Example 3.2: Consider a little adjustment of positions of treatments in a Singular PBIB design (Clatworthy, 1973) with parameters $v = 10$, $b = 10$, $r = 4$, $k = 4$, $m = 5$, $n = 2$, $n_1 = 1$, $n_2 = 8$, $\lambda_1 = 4$ & $\lambda_2 = 1$. Then PBIB design becomes a GDGNPBIB design in circular blocks having parameters ($v = 10$, $b = 10$, $r = 4$, $k = 4$, $m = 5$, $n = 2$, $n_1 = 1$, $n_2 = 8$, $\lambda_{11}=4$, $\lambda_{12}=1$, $\lambda_{21}=0$ and $\lambda_{22}=1$), Here, $2r = 8$ and $2r/(sn1+tn2) = 8/(0+8) = 1$; where $s=0$ & $t=1$, whose solution is given by:

7	1	2	6	7	1
8	2	3	7	8	2
9	3	4	8	9	3
10	4	5	9	10	4
6	5	1	10	6	5
3	6	8	1	3	6
4	7	9	2	4	7
5	8	10	3	5	8
1	9	6	4	1	9
2	10	7	5	2	10

4. EFFICIENCY OF GDGNPBIB DESIGNS FOR CORRELATED OBSERVATIONS

For a connected block design d, let $\kappa_1, \kappa_2, \ldots, \kappa_{v-1}$, be the non-zero $(v-1)$ eigenvalues of C- matrix of the design. The design is universal optimal as defined by Kifer (1975), if all κ_i's are equal with maximum trace of C-matrix. Thus, the above GDGNPBIB designs are not universally optimum like any GDPBIB designs.

Now define $\varphi_A(d) = \frac{1}{v-1}\sum_{i=1}^{v-1}\kappa_i^{-1}$ and $\varphi_D(d) = \left(\prod_{i=1}^{v-1}\kappa_i^{-1}\right)^{\frac{1}{v-1}}$.

Then, a design is A- [D-] optimal if it maximizes the $\varphi_A(d)$, $\varphi_D(d)$ over D (v, b, k). The A and D efficiencies of a design d over D (v, b, k) given by Gill and Shukla (1985), as:

$$e_A(d) = \frac{\varphi_A(d^*)}{\varphi_A(d)}$$

$$e_D(d) = \frac{\varphi_D(d^*)}{\varphi_D(d)}$$

Where d^* is a hypothetical universally optimal design whose information matrix C has nonzero eigenvalue are equal i.e., $\frac{1}{v-1}\sum_{i=1}^{v-1}\kappa_i^{-1}$ with multiplicity v-1.

Result 4.1: If $\frac{\kappa_1}{\kappa_2} = \psi$, be a constant for different values of $\rho(-1 \le \rho \le +1)$ of a GDGNPBIB design, then the efficiencies (A and D) of the design will remain constant over different values of ρ.

Proof: Let GDGNPBIB be a design with parameters $v = mn$, b, r, k, $n_1 = n$-1, $n_2 = n(m$-1), λ_{11}, λ_{12}, λ_{21} and λ_{22}; its C matrix has $\alpha_1+\alpha_2$ nonzero eigen values, let these be denoted by κ_1 and κ_2 with multiplicity α_1 and α_2, respectively. Let us consider the above definitions of optimality, φ_A and φ_D values for d and d*.

If $\frac{\kappa_1}{\kappa_2} = \psi$, $\Rightarrow$ $\kappa_1 = \psi\kappa_2$, then the efficiencies (A and D) of a design d over d* will be $e_A(d) = \left[\frac{(\alpha_1+\alpha_2)^2\psi}{(\alpha_1\psi+\alpha_2)(\alpha_1+\alpha_2\psi)}\right]$ and

$$e_D(d) = \left[\frac{(\alpha_1+\alpha_2)\left(\psi^{\frac{\alpha_1}{\alpha_1+\alpha_2}}\right)}{(\alpha_1\psi+\alpha_2)}\right].$$

When ψ is a constant for every value of ρ, then these efficiency values remain same over the different ρ values because efficiencies are the function of ø value and not the function of ρ. Actually, it is also observed that if in a GDGNPBIB design, the following relation holds, then the efficiencies (A and D) of the design will remain constant over different values of ρ.

$$\left[2(n_2\lambda_{12}^2 + n_1\lambda_{11}\lambda_{12}) - k\lambda_{11}\lambda_{22}(n_1+1) - n_2\lambda_{12}\lambda_{22}(k-1) + \lambda_{12}\lambda_{21}(n_1+k)\right] = 0$$

Remark 1: If ρ $(+0.6 \le \rho \le +1)$, the α_1 non-zero eigen roots (κ_1's) of C matrix become negative for a GDGNPBIB design with correlated observations. The efficiencies (A and D) of the designs may be negative or more than unity. In such cases, ρ values are ignored.

***Remark*2**: Two associate class PBIB designs with any association scheme for $k = 3$, will be a GNPBIB2 as in definition 1.4.

The solutions of GDGNPBIB Designs are listed in Table 1 for k(=4).Efficiency values (A and D) of Table 1 designs are presented in Table 2 for different values of ρ ($-1 \leq \rho \leq +0.5$).

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Table 1. List of GD Generalized Neighbour Partially Balanced Incomplete Block Designs in circular blocks for k=4

v	b	R	k	λ_{11}	λ_{12}	n_1	n_2	λ_{21}	λ_{22}	Adjusted Solutions		Source*
6	3	2	4	2	1	1	4	0	1	1 2 4 5 2 3 5 6 3 1 6 4		S1 (3,2)
6	6	4	4	4	2	1	4	0	2	1 2 4 5 2 3 5 6 3 1 6 4	4 5 1 2 5 6 2 3 6 4 3 1	S2 (3,2)
8	6	3	4	3	1	1	6	0	1	1 2 5 6 3 4 7 8 1 3 5 7	2 4 6 8 1 4 5 8 2 3 6 7	S7 (4,2)
10	10	4	4	4	1	1	8	0	1	1 2 6 7 2 3 7 8 3 4 8 9 4 5 9 10 5 1 10 6	6 8 1 3 7 9 2 4 8 10 3 5 9 6 4 1 10 7 5 2	S9 (5,2)
12	15	5	4	5	1	1	10	0	1	1 2 7 8 3 4 9 10 5 6 11 12 1 3 7 9 2 6 8 12 4 5 10 11 1 4 7 10 2 5 8 11	3 6 9 12 1 5 7 11 2 3 8 9 4 6 10 12 1 6 7 12 2 4 8 10 3 5 9 11	S11 (6,2)
14	21	6	4	6	1	1	12	0	1	1 2 8 9 2 3 9 10 3 4 10 11 4 5 11 12 5 6 12 13 6 7 13 14 7 1 14 8 1 3 8 10 2 4 9 11 3 5 10 12 4 6 11 13	5 7 12 14 6 1 13 8 7 2 14 9 1 4 8 11 2 5 9 12 3 6 10 13 4 7 11 14 5 1 12 8 6 2 13 9 7 3 14 10	S13 (7,2)
16	28	7	4	7	1	1	14	0	1	1 2 9 10 3 4 11 12 5 6 13 14 7 8 15 16 1 4 9 12 2 7 10 15 3 5 11 13 6 8 14 16 1 6 9 14 2 5 10 13 3 8 11 16 4 7 12 15 1 8 9 16 2 4 10 12	3 6 11 14 5 7 13 15 1 3 9 11 2 8 10 16 4 5 12 13 6 7 14 15 1 5 9 13 2 6 10 14 3 11 7 15 4 8 12 16 1 7 9 15 2 3 10 11 4 6 12 14 5 8 13 16	S14 (8,2)
6	18	12	4	9	6	2	3	6	4	1 2 4 6 2 3 5 1 3 4 6 2 4 5 1 3 5 6 2 4 6 1 3 5 1 2 6 4 2 3 1 5 3 4 2 6	4 5 3 1 5 6 4 2 6 1 5 3 2 1 4 6 3 2 5 1 4 3 6 2 5 4 1 3 6 5 2 4 1 6 3 5	R94 (2,3)

Note: *Clatworthy, 1973; Figures in parentheses indicates the association scheme (m, n)

v	b	r	k	λ11	λ12	n1	n2	λ21	λ22	Adjusted Solutions		Source*
8	30	15	4	9	6	1	6	6	4	1 5 3 4 1 5 7 8 2 6 7 4 2 6 3 8 3 7 1 2 3 7 5 6 4 8 1 6 4 8 2 5 1 5 2 6 3 7 4 8 1 5 4 3 1 5 8 7 2 6 4 7 2 6 8 3 3 7 2 1	3 7 6 5 4 8 6 1 4 8 5 2 1 5 6 2 3 7 8 4 1 3 5 4 1 7 5 8 2 7 6 4 2 3 6 8 3 1 7 2 3 5 7 6 4 1 8 6 4 2 8 5 1 2 5 6 3 4 7 8	R97 (4,2)
9	27	12	4	9	3	2	6	6	2	1 4 7 2 2 5 8 3 3 6 9 4 4 7 1 5 5 8 2 6 6 9 3 7 7 1 4 8 8 2 5 9 9 3 6 1 1 4 2 7 2 5 3 8 3 6 4 9 4 7 5 1 5 8 6 2	6 9 7 3 7 1 8 4 8 2 9 5 9 3 1 6 1 2 4 7 2 3 5 8 3 4 6 9 4 5 7 1 5 6 8 2 6 7 9 3 7 8 1 4 8 9 2 5 9 1 3 6	R104 (3,3)
10	75	30	4	15	6	4	5	10	4	1 2 3 4 1 2 5 6 1 3 5 7 1 4 8 9 2 3 8 9 2 4 7 10 3 4 6 10 1 6 8 10 1 7 9 10 2 5 8 10 2 6 7 9 3 5 9 10 3 6 7 8 4 5 6 9 4 5 7 8 1 3 5 7 1 3 5 9 1 3 7 9 1 5 7 9 3 5 7 9 2 4 6 8 2 4 6 10 2 4 8 10 2 6 8 10 4 6 8 10 1 2 4 3 1 2 6 5 1 3 7 5 1 4 9 8 2 3 9 8 2 4 10 7 3 4 10 6 1 6 10 8 1 7 10 9 2 5 10 8 2 6 9 7 3 5 10 9 3 6 8 7	4 5 9 6 4 5 8 7 1 3 7 5 1 3 9 5 1 3 9 7 1 5 9 7 3 5 9 7 2 4 8 6 2 4 10 6 2 4 10 8 2 6 10 8 4 6 10 8 1 3 2 4 1 5 2 6 1 5 3 7 1 8 4 9 2 8 3 9 2 7 4 10 3 6 4 10 1 8 6 10 1 9 7 10 2 8 5 10 2 7 6 9 3 9 5 10 3 7 6 8 4 6 5 9 4 7 5 8 1 5 3 7 1 5 3 9 1 7 3 9 1 7 5 9 3 7 5 9 2 6 4 8 2 6 4 10 2 8 4 10 2 8 6 10 4 8 6 10	R107 (2,5)

Note: *Clatworthy, 1973; Figures in parentheses indicates the association scheme (m, n)

v	b	r	k	λ_{11}	λ_{12}	n_1	n_2	λ_{21}	λ_{22}	Adjusted Solutions		Source*
12	36	12	4	6	3	1	10	4	2	1 2 5 7	7 8 1 11	R109
										2 3 6 8	8 9 2 12	(6,2)
										3 4 7 9	9 10 3 1	
										4 5 8 10	10 11 4 2	
										5 6 9 11	11 12 5 3	
										6 7 10 12	12 1 6 4	
										7 8 11 1	1 5 2 7	
										8 9 12 2	2 6 3 8	
										9 10 1 3	3 7 4 9	
										10 11 2 4	4 8 5 10	
										11 12 3 5	5 9 6 11	
										12 1 4 6	6 10 7 12	
										1 2 7 5	7 11 8 1	
										2 3 8 6	8 12 9 2	
										3 4 9 7	9 1 10 3	
										4 5 10 8	10 2 11 4	
										5 6 11 9	11 3 12 5	
										6 7 12 10	12 4 1 6	
6	9	6	4	3	4	2	3	0	4	1 2 3 4	3 6 5 2	SR35
										5 6 1 2	1 6 3 2	(2,3)
										3 4 5 6	5 4 1 6	
										1 4 3 6	3 2 5 4	
										5 2 1 4		
8	24	12	4	0	6	1	6	0	4	1 2 3 4	3 8 6 1	SR36
										5 6 7 8	7 4 2 5	(4,2)
										2 7 8 1	4 1 7 6	
										6 3 4 5	8 5 3 2	
										3 8 1 6	2 1 3 4	
										7 4 5 2	6 5 7 8	
										4 1 6 7	7 2 8 1	
										8 5 2 3	3 6 4 5	
										1 2 4 3	8 3 1 6	
										5 6 8 7	4 7 5 2	
										2 7 1 8	1 4 6 7	
										6 3 5 4	5 8 2 3	
12	27	9	4	0	3	2	9	0	2	1 2 3 4	10 9 8 3	SR41
										7 10 5 4	1 11 12 10	(4,3)
										6 11 9 4	9 2 12 7	
										1 7 6 8	5 3 12 6	
										11 5 2 8	1 3 2 4	
										10 9 3 8	7 5 10 4	
										1 11 10 12	6 9 11 4	
										9 2 7 12	1 6 7 8	
										5 3 6 12	11 2 5 8	
										1 2 4 3	10 3 9 8	
										7 10 4 5	1 10 11 12	
										6 11 4 9	9 7 2 12	
										1 7 8 6	5 6 3 12	
										11 5 8 2		

Note: *Clatworthy, 1973; Figures in parentheses indicates the association scheme (m, n)

v	b	r	k	λ_{11}	λ_{12}	n_1	n_2	λ_{21}	λ_{22}	Adjusted Solutions		Source*
16	48	12	4	0	3	3	12	0	2	1 2 3 4	1 7 14 12	SR44
										5 6 7 8	2 8 13 11	(4,4)
										9 10 11 12	3 5 16 10	
										13 14 15 16	4 6 15 9	
										1 6 11 16]	1 8 15 10	
										2 5 12 15	2 7 16 9	
										3 8 9 14	3 6 13 12	
										4 7 10 13	4 5 14 11	
										1 7 12 14	1 3 2 4	
										2 8 11 13	5 7 6 8	
										3 5 10 16	9 11 10 12	
										4 6 9 15	13 15 14 16	
										1 8 10 15	1 11 6 16	
										2 7 9 16	2 12 5 15	
										3 6 12 13	3 9 8 14	
										4 5 11 14	4 10 7 13	
										1 2 4 3	1 12 7 14	
										5 6 8 7	2 11 8 13	
										9 10 12 11	3 10 5 16	
										13 14 16 15	4 9 6 15	
										1 6 16 11	1 10 8 15	
										2 5 15 12	2 9 7 16	
										3 8 14 9	3 12 6 13	
										4 7 13 10	4 11 5 14	

Note: *Clatworthy, 1973; Figures in parentheses indicates the association scheme (m, n)

Table 2. A- and D- efficiency of GD Generalized Neighbour Partially Balanced Incomplete block design of series of method 3.2 for correlated observations

v	b	r	k	λ_{11}	λ_{12}	m	n	n_1	n_2	λ_{21}	λ_{22}	ρ	A-efficiency	D-efficiency
6	3	2	4	2	1	3	2	1	4	0	1	-1	0.857	0.922
												-0.9	0.879	0.934
												-0.8	0.900	0.947
												-0.7	0.921	0.958
												-0.6	0.942	0.970
												-0.5	0.963	0.981
												-0.4	0.979	0.989
												-0.3	0.992	0.996
												-0.2	0.999	0.999
												-0.1	0.997	0.999
												0	0.980	0.990
												0.1	0.940	0.970
												0.2	0.861	0.932
												0.3	0.718	0.858
												0.4	0.464	0.709
												0.5	0.061	0.308

v	b	r	k	λ_{11}	λ_{12}	m	n	n_1	n_2	λ_{21}	λ_{22}	ρ	A-efficiency	D-efficiency
6	6	4	4	4	2	3	2	1	4	0	2	-1	0.857	0.922
												-0.9	0.879	0.934
												-0.8	0.900	0.947
												-0.7	0.921	0.958
												-0.6	0.942	0.970
												-0.5	0.963	0.981
												-0.4	0.979	0.989
												-0.3	0.992	0.996
												-0.2	0.999	0.999
												-0.1	0.997	0.999
												0	0.980	0.990
												0.1	0.940	0.970
												0.2	0.861	0.932
												0.3	0.718	0.858
												0.4	0.464	0.709
												0.5	0.061	0.308
8	6	3	4	3	1	4	2	1	6	0	1	-1	0.891	0.942
												-0.9	0.910	0.953
												-0.8	0.929	0.963
												-0.7	0.948	0.973
												-0.6	0.965	0.982
												-0.5	0.981	0.991
												-0.4	0.992	0.996
												-0.3	0.999	0.999
												-0.2	0.999	0.999
												-0.1	0.988	0.994
												0	0.961	0.981
												0.1	0.909	0.955
												0.2	0.819	0.909
												0.3	0.669	0.828
												0.4	0.420	0.671
												0.5	0.053	0.272
10	10	4	4	4	1	5	2	1	8	0	1	-1	0.908	0.952
												-0.9	0.926	0.962
												-0.8	0.944	0.971
												-0.7	0.960	0.980
												-0.6	0.975	0.987
												-0.5	0.989	0.994
												-0.4	0.997	0.998
												-0.3	0.999	0.999
												-0.2	0.996	0.998
												-0.1	0.980	0.990
												0	0.947	0.974
												0.1	0.890	0.945
												0.2	0.795	0.895
												0.3	0.643	0.810
												0.4	0.398	0.649
												0.5	0.049	0.254

v	b	r	k	λ_{11}	λ_{12}	m	n	n_1	n_2	λ_{21}	λ_{22}	ρ	A-efficiency	D-efficiency
12	15	5	4	5	1	6	2	1	10	0	1	-1	0.919	0.958
												-0.9	0.936	0.967
												-0.8	0.953	0.976
												-0.7	0.968	0.984
												-0.6	0.981	0.990
												-0.5	0.993	0.996
												-0.4	0.999	0.999
												-0.3	0.999	0.999
												-0.2	0.993	0.996
												-0.1	0.974	0.987
												0	0.938	0.969
												0.1	0.877	0.938
												0.2	0.780	0.886
												0.3	0.626	0.799
												0.4	0.385	0.636
												0.5	0.047	0.243
14	21	6	4	6	1	7	2	1	12	0	1	-1	0.926	0.962
												-0.9	0.942	0.970
												-0.8	0.958	0.979
												-0.7	0.972	0.986
												-0.6	0.985	0.992
												-0.5	0.995	0.997
												-0.4	0.999	0.999
												-0.3	0.999	0.999
												-0.2	0.990	0.995
												-0.1	0.969	0.985
												0	0.931	0.965
												0.1	0.868	0.933
												0.2	0.769	0.880
												0.3	0.615	0.791
												0.4	0.376	0.626
												0.5	0.046	0.236
16	28	7	4	7	1	8	2	1	14	0	1	-1	0.931	0.964
												-0.9	0.947	0.973
												-0.8	0.962	0.981
												-0.7	0.976	0.988
												-0.6	0.987	0.993
												-0.5	0.996	0.998
												-0.4	0.999	0.999
												-0.3	0.998	0.999
												-0.2	0.988	0.994
												-0.1	0.965	0.983
												0	0.926	0.963
												0.1	0.862	0.929
												0.2	0.761	0.875
												0.3	0.607	0.785
												0.4	0.369	0.619
												0.5	0.045	0.230

v	b	r	k	λ_{11}	λ_{12}	m	n	n_1	n_2	λ_{21}	λ_{22}	ρ	A-efficiency	D-efficiency
6	9	6	4	3	4	2	3	2	3	0	4	-1	0.857	0.913
												-0.9	0.869	0.921
												-0.8	0.883	0.929
												-0.7	0.896	0.938
												-0.6	0.911	0.948
												-0.5	0.927	0.958
												-0.4	0.942	0.967
												-0.3	0.957	0.976
												-0.2	0.973	0.985
												-0.1	0.987	0.993
												0	0.997	0.999
												0.1	0.999	1.000
												0.2	0.984	0.992
												0.3	0.926	0.967
												0.4	0.746	0.896
												0.5	0.149	0.601
$^{\$}6$	18	12	4	9	6	2	3	2	3	6	4	0	0.992	0.996
$^{\$}8$	30	15	4	9	6	4	2	1	6	6	4	0	0.997	0.998
$^{\$}9$	27	12	4	9	3	3	3	2	6	6	2	0	0.952	0.978
$^{\$}10$	75	30	4	15	6	2	5	4	5	10	4	0	0.969	0.987
$^{\$}12$	36	12	4	6	3	6	2	1	10	4	2	0	0.994	0.997
$^{\$}8$	24	12	4	0	6	4	2	1	6	0	4	0	0.980	0.990
$^{\$}12$	27	9	4	0	3	4	3	2	9	0	2	0	0.984	0.991
$^{\$}16$	48	12	4	0	3	4	4	3	12	0	2	0	0.987	0.993

Note: $ indicates efficiencies (A & D) are independent to ρ values ($+1 \leq \rho \leq -1$), see also Result 4.1.